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On the Isomorphism of a Quantum Logic with the Logic of the Projections in a Hilbert Space. II

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Abstract

The results about the isomorphism of a quantum logic $\mathscr{L}$ with the logic of the projections in a separable Hilbert space previously obtained with the introduction of the topology of states are completed, including the case of non-separable Hilbert space, and showing that the continuity of the antiautomorphism θ of the division ring $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ determined by $\mathscr{L}$ follows from the general topological assumptions on $\mathscr{L}$.

1. *Introduction*

The introduction in a logic $\mathscr{L}$ (a σ-complete, orthocomplemented and weakly modular lattice) of the so-called *topology of states* (see the Appendix) allowed us, in a preceding paper (Cirelli & Cotta-Ramusino, 1973), to formulate conditions under which the division ring determined by $\mathscr{L}$ is the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the quaternion division ring $\mathbb{Q}$. The main result we obtained can be summarised in the following theorem (Cirelli & Cotta-Ramusino, 1973, Theorem 5.2).

Let $\mathscr{L}$ be a logic and let $\mathscr{L}$ be endowed with the topology of states. Then:

(1) if $\mathscr{L}$ is a projective logic such that every family of mutually orthogonal points is at most countable and conditions $\mathscr{L}_1$–$\mathscr{L}_5$ below are satisfied, then $\mathscr{L}$ is isomorphic to the projective logic $\mathscr{L}(V, \langle .,. \rangle)$ of all linear manifolds closed relative to the θ-bilinear form $\langle .,. \rangle$, where V is a (left) linear space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ with dim $V \geqslant 4$;

(2) if in addition the antiautomorphism θ of the division ring $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ is continuous then V is a separable Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ respectively.

Conversely, if $\mathscr{L}$ is isomorphic to the logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ of the projections in a separable Hilbert space $\mathscr{H}$ over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with dim $\mathscr{H} \geqslant 4$, then $\mathscr{L}$ is a projective logic such that every family of mutually orthogonal points is at most countable, conditions $\mathscr{L}_1$-$\mathscr{L}_5$ are satisfied and the automorphism θ is continuous.

Conditions $\mathscr{L}_1$-$\mathscr{L}_5$ are the following:

($\mathscr{L}_1$) $s(a) = s(b)$ for every pure state s implies $a = b$, namely the set $\mathscr{P}$ of pure states is separating,

($\mathscr{L}_2$) for any finite element a of $\mathscr{L}$, $\mathscr{L}[0, a]$ is a compact subset of $\mathscr{L}$,

($\mathscr{L}_3$) $\mathscr{L}$ is second countable,

($\mathscr{L}_4$) for any line l of $\mathscr{L}$ the set of all points of ℓ but one arbitrary chosen is a connected set,

($\mathscr{L}_5$) no plane of $\mathscr{L}$ is trivial, for any plane u of $\mathscr{L}$ the intersection point of two lines in u is a continuous function of the two lines and the union line of two points in u is a continuous function of the two points.

In this paper we will enlarge these results in two respects: first we shall drop from the theorem the condition of separability of the Hilbert space, second we shall show that the continuity of the antiautomorphism θ follows from the general topological assumptions on $\mathscr{L}$. Precisely we shall show that the following theorem holds.

Theorem 1.1. Let $\mathscr{L}$ be a logic and let $\mathscr{L}$ be endowed with the topology of states. A necessary and sufficient condition in order that $\mathscr{L}$ be isomorphic to the logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ of the projections in a Hilbert space $\mathscr{H}$ over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with $\dim \mathscr{H} \geqslant 4$ is that $\mathscr{L}$ be a complete projective logic satisfying conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$ below. Moreover the Hilbert space $\mathscr{H}$ is separable if and only if $\mathscr{L}$ is such that every family of mutually orthogonal points is at most countable.

Conditions $\mathscr{L}'_1$, $\mathscr{L}'_2$, $\mathscr{L}'_4$ and $\mathscr{L}'_5$ are the same as $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_4$ and $\mathscr{L}_5$ respectively while condition $\mathscr{L}'_3$ reads as follows:

($\mathscr{L}'_3$) for every finite element a of $\mathscr{L}$, $\mathscr{L}[0, a]$ is second countable.

2. *Proof of the Theorem*

Let $\mathscr{H}$ be a Hilbert space over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with $\dim \mathscr{H} \geqslant 4$. Then $\mathscr{L}(\mathscr{H}, \mathbb{D})$ is a complete projective logic (Varadarajan, 1968, Theorem 7.40). Moreover from the Gleason theorem† it follows that the topology of states in $\mathscr{L}(\mathscr{H}, \mathbb{D})$ coincides with the induced weak operator topology (Cirelli & Cotta-Ramusino, 1973, Theorem 4.1). On account of this we have immediately that $\mathscr{L}(\mathscr{H}, \mathbb{D})$ endowed with the topology of states satisfies condition $\mathscr{L}'_1$ and we can proceed exactly in the same way as in Cirelli & Cotta-Ramusino (1973, Section 3) to prove that $\mathscr{L}(\mathscr{H}, \mathbb{D})$ satisfies conditions $\mathscr{L}'_4$ and $\mathscr{L}'_5$, while the fact that $\mathscr{L}(\mathscr{H}, \mathbb{D})$ satisfies conditions $\mathscr{L}'_2$ and $\mathscr{L}'_3$ follows from the following lemma which is a slight modification of Theorem 3.1 in Cirelli & Cotta-Ramusino (1973).

† The Gleason theorem holds also for non-separable Hilbert spaces. We are very much indebted to Prof. M. Guenin for a private communication on this extension of the Gleason theorem.

Lemma 2.1. Let Q be a projection of finite rank. Then the geometry $\mathscr{L}[0, Q]$ is a second countable compact subset of $\mathscr{L}(\mathscr{H}, \mathbb{D})$.†

Let now $\mathscr{L}$ be a logic endowed with the topology of states and let $\mathscr{L}$ be isomorphic to $\mathscr{L}(\mathscr{H}, \mathbb{D})$. Then from Theorem A.1 in the Appendix we have that the topology of states in $\mathscr{L}$ coincides with the topology transferred from $\mathscr{L}(\mathscr{H}, \mathbb{D})$ by the isomorphism. Therefore the logic $\mathscr{L}$ is a complete projective logic satisfying conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$.

Conversely, let $\mathscr{L}$ be a complete projective logic satisfying conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$. From Varadarajan (1968, Theorem 7.40) and from Cirelli & Cotta-Ramusino (1973, Section 5) we have that $\mathscr{L}$ is isomorphic to the logic $\mathscr{L}(V, \langle .,. \rangle)$, where V is a vector space over $\mathbb{D}(\mathbb{R}, \mathbb{C}$ or $\mathbb{Q})$ with dim $V \geqslant 4$ and $\langle .,. \rangle$ is a θ-bilinear form on $V \times V$ related to the antiautomorphism θ induced by the logic on $\mathbb{D}$. Indeed the proof of Theorem 5.1 in Cirelli & Cotta-Ramusino (1973) is still valid if one substitutes conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$ for conditions $\mathscr{L}_1$-$\mathscr{L}_5$ and requires the completeness of the logic instead of the property that every family of mutually orthogonal points is at most countable.

If we now admit that conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$ imply that the antiautomorphism θ is continuous, then $\langle .,. \rangle$ is an inner product and, on account of Varadarajan (1968, Lemma 7.42), which ensures us the completeness of the space V, we have that the logic $\mathscr{L}$ is isomorphic to a logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$. Moreover, it is obvious that $\mathscr{H}$ is separable if and only if the logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ has the property that every family of mutually orthogonal points is at most countable.

Thus to have the complete proof of the theorem we have only to show that conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$ imply the continuity of the antiautomorphism θ.

3. *Continuity of the Antiautomorphism* θ

Let $\mathscr{L}$ be a complete projective logic which satisfies conditions $\mathscr{L}'_1$-$\mathscr{L}'_5$. As we have seen in Section 2, $\mathscr{L}$ is isomorphic to the projective logic $\mathscr{L}(V, \langle .,. \rangle)$ of all $\langle .,. \rangle$-closed linear manifolds of a linear space V over $\mathbb{D}(\mathbb{R}, \mathbb{C}$ or $\mathbb{Q})$ with dim $V \geqslant 4$. The isomorphism $\zeta : \mathscr{L} \to \mathscr{L}(V, \langle .,. \rangle$ is constructed in the following way.

Let $\mathscr{L}' := \{a \in \mathscr{L} \mid a \text{ finite}\}$; $\mathscr{L}'$ is a generalised geometry eventually of infinite dimension. Let (O, P_j), $j \in J$, be a frame at O in $\mathscr{L}'$. If for any $j \in J$ we fix a point E_j on the axis $m_j = O \vee P_j$ distinct from O and P_j we may construct the division ring $\mathbb{D}_j = \mathbb{D}_j(O, E_j, P_j)$ on m_j with O, E_j, P_j as origin, unit

† Proof of Lemma 2.1:

Let K = Range (Q) and let $\mathscr{V}_Q$ be the linear manifold in $\mathscr{B}(\mathscr{H})$ (the algebra of all bounded operators on $\mathscr{H}$) generated by the elements of $\mathscr{L}[0, Q]$. To every operator $A \in \mathscr{V}_Q$ we can associate its restriction $\hat{A}$ to K. Obviously $\hat{A}$ belongs to $\mathscr{B}(K)$ and the correspondence $A \rightsquigarrow \hat{A}$ from $\mathscr{V}_Q$ into $\mathscr{B}(K)$ can be easily shown to be linear and injective. Therefore $\mathscr{V}_Q$ is a finite dimensional linear manifold of $\mathscr{B}(\mathscr{H})$. Then on $\mathscr{V}_Q$ the induced weak, strong and uniform topologies coincide with the 'euclidean' topology. Considering $\mathscr{L}[0, Q]$ as a subset of $\mathscr{V}_Q$ one has immediately that $\mathscr{L}[0, Q]$ is second countable and bounded; moreover, essentially by the same arguments as in the proof of Theorem 3.1 in Cirelli & Cotta-Ramusino (1973), it can be proved that it is closed in $\mathscr{V}_Q$. Therefore $\mathscr{L}[0, Q]$ is second countable and compact.

point and point at infinity respectively. All the division rings $\mathbb{D}_j$ are isomorphic and there exist a division ring $\mathbb{D}$ and a set of isomorphisms φ_j of $\mathbb{D}_j$ onto $\mathbb{D}$ such that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{D}_i & \xrightarrow{\varphi_{ij}} & \mathbb{D}_j \\ & {\scriptstyle\varphi_i}\searrow \quad \swarrow{\scriptstyle\varphi_j} & \\ & \mathbb{D} & \end{array} \tag{3.1}$$

where the isomorphism φ_{ij} of $\mathbb{D}_i$ with $\mathbb{D}_j$ is given by $\varphi_{ij}(X) = (X \vee P_{ij}) \wedge m_j$ (P_{ij} is the intersection point of the lines $E_i \vee E_j$ and $P_i \vee P_j$).

To every point $Q \in \mathscr{L}'$ not lying at infinity (in symbols: $Q \nless \mathscr{L}'_\infty$), that is such that, for every finite subset K of J,

$$Q \nless u(K)$$

where

$$u(K) = \begin{cases} 0 & \text{if } K = \emptyset \text{ (the void set)} \\ \bigvee_{j \in K} P_j & \text{if } K \neq \emptyset, \end{cases}$$

we can associate the set of points $\{M_j^Q\}$, $j \in J$, with M_j^Q given by

$$\left.\begin{array}{lll} M_j^Q = 0, & \forall_j \in J, & \text{if } Q = 0 \\ M_j^Q = (Q \vee u(K - \{j\})) \wedge m_j, & & \text{if } Q \neq 0, j \in K \\ M_j^Q = 0, & \text{if } Q \neq 0, & j \notin K, \end{array}\right\} \tag{3.2}$$

where K is any finite subset of J such that $Q < O \vee u(K)$ (M_j^Q, $j \in J$, does not depend on the choice of such a subset K (Varadarajan, 1968, Lemma 5.18)). One has obviously

$$M_j^Q \in \mathbb{D}_j, \qquad j \in J \tag{3.3}$$

Let now V be the (left) free linear space over $\mathbb{D}$ generated by $J \cup \{\infty\}$, where ∞ is one more element added to the set of indices J. To every point $Q \in \mathscr{L}'$ such that $Q \nless \mathscr{L}'_\infty$ we can associate a vector $g^Q \in V$ in the following way

$$\begin{aligned} g^Q(j) &= \varphi_j(M_j^Q), \qquad j \in J \\ g^Q(\infty) &= 1 \end{aligned} \tag{3.4}$$

where 1 is the unit element of $\mathbb{D}$. To shorten the notation we shall write $g^Q = \{\varphi_j(M_j^Q), 1\}$. If on the contrary Q is a point belonging to $\mathscr{L}'_\infty$ we can choose a point $Q' < O \vee Q$ such that $Q' \neq Q$ and $Q' \neq O$. Then $Q' \nless \mathscr{L}'_\infty$ and we can associate to Q the vector.

$$g^{Q,Q'} = \{\varphi_j(M_j^Q), 0\} \tag{3.5}$$

where 0 is the zero element of $\mathbb{D}$.

If $\mathscr{L}(V, \mathbb{D})$ is the generalised geometry of all finite dimensional subspaces of V, then

$$Q \longrightarrow \gamma(Q) := \begin{cases} \mathbb{D}.g^Q, & \text{if } Q \not< \mathscr{L}'_\infty \\ \mathbb{D}.g^{Q,Q'}, & \text{if } Q < \mathscr{L}'_\infty \end{cases} \tag{3.6}$$

is a one-one collinearity preserving map of the set of all the points of the generalised geometry $\mathscr{L}'$ onto the set of all points of the generalised geometry $\mathscr{L}(V, \mathbb{D})$ (Varadarajan, 1968, Lemma 5.25) (remark that $\mathbb{D}.g^{QQ'}$ does not depend on the choice of the point Q').

The desired isomorphism $\zeta : \mathscr{L} \to \mathscr{L}(V, \langle .\,,.\rangle)$ is given by

$$a \longrightarrow \zeta(a) := \{x \in V \mid x \in \gamma(P) \text{ for some point } P < a\} \tag{3.7}$$

Let now a_n be a fixed finite element of $\mathscr{L}$ such that $\dim a_n$ $(= \dim \mathscr{L}[\mathbf{0}, a_n]) = n \geqslant 4$. Henceforth it will be understood that the frame (O, P_j) is an 'adapted' one to $\mathscr{L}[\mathbf{0}, a_n]$. This simply means that O and $n - 1$ out of the P_j belong to $\mathscr{L}[\mathbf{0}, a_n]$ (these $n - 1$ points will be denoted by $P_1, P_2, \ldots, P_{n-1}$). The restriction to $\mathscr{L}[\mathbf{0}, a_n]$ of the isomorphism ζ will be called ξ. Under the ordering inherited from $\mathscr{L}$ and the orthocomplementation

$$^+ : \mathscr{L}[\mathbf{0}, a_n] \to \mathscr{L}[\mathbf{0}, a_n], \qquad b \rightsquigarrow b^+ := b^* \wedge a_n \tag{3.8}$$

where b^* is the orthocomplementation of b in $\mathscr{L}$, $\mathscr{L}[\mathbf{0}, a_n]$ is a logic. On $\mathscr{L}(V_n, \mathbb{D})$, where V_n is the n-dimensional linear space over $\mathbb{D}$ given by $V_n = \xi(a_n)$ the map

$$^\perp : \mathscr{L}(V_n, \mathbb{D}) \to \mathscr{L}(V_n, \mathbb{D}), \qquad B = \xi(b) \rightsquigarrow B^\perp := \xi(b^+) \tag{3.9}$$

is an orthocomplementation. Thus $\mathscr{L}(V_n, \mathbb{D})$ is a logic and ξ is an isomorphism of $\mathscr{L}[\mathbf{0}, a_n]$ with $\mathscr{L}(V_n, \mathbb{D})$.

From Theorem A.1 of the Appendix the isomorphism ζ is also a homeomorphism when on $\mathscr{L}$ and $\mathscr{L}(V, \langle .\,,.\rangle)$ we introduce the topologies of states. If on $\mathscr{L}[\mathbf{0}, a_n]$ and on $\mathscr{L}(V_n, \mathbb{D})$ we consider the induced topologies, ξ as well as a homeomorphism.

We now proceed to the study of the antiautomorphism θ of $\mathbb{D}$ associated to the θ-bilinear form $\langle .\,,.\rangle$ on $V \times V$. Let $\mathbb{D}^0$ be the division ring dual to $\mathbb{D}$, V_n^* the space dual to V_n and $\mathscr{L}(V_n^*, \mathbb{D}^0)$ the lattice of subspaces of V_n^* (note that V_n^* is considered as an n-dimensional linear space over $\mathbb{D}^0$). We introduce the maps

$$^0 : \mathscr{L}(V_n, \mathbb{D}) \to \mathscr{L}(V_n^*, \mathbb{D}^0), \qquad M \rightsquigarrow M^0 := \{\lambda \in V_n^* \mid \lambda(x) = 0, \forall x \in M\} \tag{3.10}$$

and

$$\eta : \mathscr{L}(V_n, \mathbb{D}) \to \mathscr{L}(V_n^*, \mathbb{D}^0), \qquad M \rightsquigarrow \eta(M) := (M^\perp)^0 \tag{3.11}$$

One can verify that η is an isomorphism between geometries. On $\mathscr{L}(V_n^*, \mathbb{D}^0)$ we consider the quotient topology relative to η and to the topology of $\mathscr{L}(V_n,$

$\mathbb{D}$), then the induced topology on $\mathbb{D}^0$ (considered as a subset of a certain line of $\mathscr{L}(V_n^*, \mathbb{D}^0)$) is the euclidean topology, that is the same topology as on $\mathbb{D}$.

Now we define a relation between the vectors of V_n and V_n^*. We say that $x \in V_n$ is related to $\tilde{x} \in V_n^*$, and write $x \sim \tilde{x}$, if $x \neq 0, \tilde{x} \neq 0$ and $\eta(\mathbb{D}.x) = \mathbb{D}^0.\tilde{x}$. From Varadarajan (1968, Lemma 3.2) we know that if $x \in V_n$ and $\tilde{x} \in V_n^*$ are such that $x \sim \tilde{x}$, then for any $y \in V_n$ such that $y \neq 0$ and $\mathbb{D}.y \neq \mathbb{D}.x$ there exists a unique $\tilde{y} \in V_n^*$ such that $y \sim \tilde{y}$ and $x - y \sim \tilde{x} - \tilde{y}$. Hence for every pair $(x, \tilde{x})$ such that $x \sim \tilde{x}$ the following map is well defined

$$T_{x,\tilde{x}} : V_n - \mathbb{D}.x \to V_n^*; \qquad T_{x,\tilde{x}}(y) = \tilde{y} \tag{3.12}$$

where $\tilde{y}$ is the unique vector such that $y \sim \tilde{y}$ and $x - y \sim \tilde{x} - \tilde{y}$.

We want to construct explicitly such a vector $\tilde{y}$ given any $x \in V_n, x \neq 0$, and a related vector $\tilde{x} \in V_n^*$ chosen in a way suitable for our purposes.

We set $x = \{x_i, x_n\}, y = \{y_i, y_n\}, i \in \mathscr{T} = \{1, 2, \ldots, n-1\}$ and suppose $x_n \neq 0$, $y_n \neq 0$ and $x_n \neq y_n$ (this is not a restriction because, with a suitable change of coordinates, we can always have this situation).

We put also

$$g^x = x_n^{-1}x = \{g_i^x, 1\}, \qquad i \in \mathscr{T} \tag{3.13}$$

$$g^y = y_n^{-1}y = \{g_i^y, 1\}, \qquad i \in \mathscr{T} \tag{3.14}$$

and consider the points $X = \xi^{-1}(\mathbb{D}.x) = \xi^{-1}(\mathbb{D}.g^x)$ and $Y = \xi^{-1}(\mathbb{D}.y) = \xi^{-1}(\mathbb{D}.g^y)$ of $\mathscr{L}[\mathbf{0}, a_n]$. Obviously these points do not lie at infinity and we have, introducing their 'coordinates' M_i^X and M_i^Y (see (3.2)),

$$g_i^x = \varphi_i(M_i^X), \qquad g_i^y = \varphi_i(M_i^Y), \qquad i \in \mathscr{T} \tag{3.15}$$

Since η is an isomorphism, defining

$$\tilde{O} = (\eta \circ \xi)(O), \qquad \tilde{P}_i = (\eta \circ \xi)(P_i), \qquad i \in \mathscr{T}$$

we have that $(\tilde{O}, \tilde{P}_i)$ is a frame at $\tilde{O}$ in $\mathscr{L}(V_n^*, \mathbb{D}^0)$ such that the points $\tilde{X} = (\eta \circ \xi)(X)$ and $\tilde{Y} = (\eta \circ \xi)(Y)$ do not lie at infinity. The axes of this frame are $\tilde{m}_i = \tilde{O} \vee \tilde{P}_i = (\eta \circ \xi)(m_i)$ and on these lines we can construct the division rings $\tilde{\mathbb{D}}_i = \tilde{\mathbb{D}}_i(\tilde{O}, \tilde{E}_i, \tilde{P}_i)$ with $\tilde{O}, \tilde{E}_i = (\eta \circ \xi)(E_i)$ and $\tilde{P}_i$ as origin, unit point and point at infinity respectively. Moreover between these division rings and the division ring $\mathbb{D}^0$ there exist isomorphisms such that the following diagrams are commutative

$$\begin{array}{ccc} \tilde{\mathbb{D}} & \xrightarrow{\tilde{\varphi}_{ij}} & \tilde{\mathbb{D}} \\ & {\scriptstyle \tilde{\varphi}_i} \searrow \quad \swarrow {\scriptstyle \tilde{\varphi}_j} & \\ & \mathbb{D}^0 & \end{array} \tag{3.16}$$

All the $\tilde{\varphi}_i$ and $\tilde{\varphi}_{ij}$ are suitable projectivities, therefore also homeomorphisms. In the same way as in (3.2) we associate to $\tilde{X}$ and $\tilde{Y}$ their 'coordinates' $\tilde{M}_i^{\tilde{X}} \in \tilde{\mathbb{D}}_i$ and $\tilde{M}_i^{\tilde{Y}} \in \tilde{\mathbb{D}}_i$ with respect to the frame $(\tilde{O}, \tilde{P}_i)$ and obviously we have

$$\tilde{M}_i^{\tilde{X}} = (\eta \circ \xi)(M_i^X), \qquad \tilde{M}_i^{\tilde{Y}} = (\eta \circ \xi)(M_i^Y), \qquad i \in \mathscr{T} \tag{3.17}$$

Now, as in (3.4), we define the vectors $\tilde{g}^{\tilde{X}} \in V_n^*$ and $\tilde{g}^{\tilde{Y}} \in V_n^*$ setting

$$\tilde{g}^{\tilde{X}} = \{\tilde{\varphi}_i(\tilde{M}_i^{\tilde{X}}), 1^0\}, \qquad \tilde{g}^{\tilde{Y}} = \{\tilde{\varphi}_i(M_i^{\tilde{Y}}), 1^0\} \tag{3.18}$$

where 1^0 is the unit element of $\mathbb{D}^0$. From Varadarajan (1968, Lemma 5.25) we have that

$$\tilde{X} = \tilde{\gamma}(\mathbb{D}^0 . \tilde{g}^{\tilde{X}}), \qquad \tilde{Y} = \tilde{\gamma}(\mathbb{D}^0 . \tilde{g}^{\tilde{Y}}) \tag{3.19}$$

where $\tilde{\gamma}$ is a one-one collinearity preserving map of the set of all the points of $\mathscr{L}(V_n^*, \mathbb{D}^0)$ onto itself, namely an element of the projective group $PGL(V_n^*)$ (MacLane & Birkhoff, 1970, Chap. XII). Hence, in virtue of a very well-known theorem of projective geometry (MacLane & Birkhoff, 1970, Chap. XII, Theorem 17) we can write

$$\tilde{\gamma}(\mathbb{D}^0 . \tilde{g}^{\tilde{X}}) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{X}}), \qquad \tilde{\gamma}(\mathbb{D}^0 . \tilde{g}^{\tilde{Y}}) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) \tag{3.20}$$

where $\tilde{\Gamma}$ is an element of the group $GL(V_n^*)$ of the automorphisms of V_n^* determined by $\tilde{\gamma}$ up to a non-zero scalar multiple of the identity automorphism of V_n^*.

Let now $z = x - y$ and set

$$g^z = z_n^{-1} z = (x_n - y_n)^{-1}(x - y) \tag{3.21}$$

The point $Z = \xi^{-1}(\mathbb{D} . z) = \xi^{-1}(\mathbb{D} . g^z)$ does not lie at infinity and we have, introducing its 'coordinates' M_i^Z,

$$g_i^z = \varphi_i(M_i^Z), \qquad i \in \mathscr{T} \tag{3.22}$$

Between the 'coordinates' M_i^Z of Z and $\tilde{M}_i^{\tilde{Z}}$ of $\tilde{Z} = (\eta \circ \xi)(Z)$ the relation

$$\tilde{M}_i^{\tilde{Z}} = (\eta \circ \xi)(M_i^Z), \qquad i \in \mathscr{T} \tag{3.23}$$

holds and defining the vector $\tilde{g}^{\tilde{Z}} \in V_n^*$ by

$$\tilde{g}^{\tilde{Z}} = \{\tilde{\varphi}_i(\tilde{M}_i^{\tilde{Z}}), 1^0\} \tag{3.24}$$

we can write, as above,

$$\tilde{Z} = \tilde{\gamma}(\mathbb{D}^0 . \tilde{g}^{\tilde{Z}}) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{Z}}) \tag{3.25}$$

Now

$$g_i^Z = (x_n - y_n)^{-1}(x_i - y_i) = (x_n - y_n)^{-1} x_n g_i^x - (x_n - y_n)^{-1} y_n g_i^y, \qquad i \in \mathscr{T} \tag{3.26}$$

Since the operations on the division rings $\mathbb{D}_i$ and $\tilde{\mathbb{D}}_i$ are defined by projectivities (Varadarajan, 1968, Chap. V) and η and ξ are isomorphisms, taking into account (3.15), (3.17), (3.18), (3.22), (3.23) and (3.24), from (3.26) we obtain

$$\tilde{g}_i^{\tilde{Z}} = (\rho(x_n) - \rho(y_n))^{-1} \rho(x_n) \tilde{g}_i^{\tilde{X}} - (\rho(x_n) - \rho(y_n))^{-1} \rho(y_n) \tilde{g}_i^{\tilde{Y}}, \quad i \in \mathscr{T} \tag{3.27}$$

where

$$\rho = \tilde{\varphi}_s \circ \eta \circ \xi \circ \tilde{\varphi}_s^{-1} \tag{3.28}$$

with s any one index belonging to $\mathscr{T}$.

From (3.20) and (3.25) we have

$$\eta(\mathbb{D}.x) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{X}}) \tag{3.29}$$

$$\eta(\mathbb{D}.y) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) \tag{3.30}$$

$$\eta(\mathbb{D}.(x-y)) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{Z}}) \tag{3.31}$$

From (3.29) we infer that $\tilde{x} \sim \tilde{\Gamma}(\tilde{g}^{\tilde{X}})$. We choose as a related vector to x exactly $\tilde{\Gamma}(\tilde{g}^{\tilde{X}})$ and look for the unique $\tilde{y}$ such that $y \sim \tilde{y}$ and $x - y \sim \tilde{\Gamma}(\tilde{g}^{\tilde{X}}) - \tilde{y}$. Since η is a lattice isomorphism and $\mathbb{D}.x \neq \mathbb{D}y$ it follows that for every non-zero vector $\tilde{y}' \in \eta(\mathbb{D}.y)$ there exist a, $b \in \mathbb{D}^0$, $a \neq 0$, $b \neq 0$, such that

$$\eta(\mathbb{D}.(x-y)) = \mathbb{D}^0 . (a\tilde{\Gamma}(\tilde{g}^{\tilde{X}}) + b\tilde{y}') \tag{3.32}$$

If we take as $\tilde{y}'$ the vector $\tilde{\Gamma}(\tilde{g}^{\tilde{Y}})$ (see (3.30)) from (3.32) and (3.31) we have

$$\mathbb{D}^0 . (a\tilde{\Gamma}(\tilde{g}^{\tilde{X}}) + b\tilde{\Gamma}(\tilde{g}^{\tilde{Y}})) = \mathbb{D}^0 . \tilde{\Gamma}(\tilde{g}^{\tilde{Z}}) \tag{3.33}$$

Obviously we can choose a, b such that $a + b = 1^0$. Then we obtain

$$a\tilde{g}_i^{\tilde{X}} + b\tilde{g}_i^{\tilde{Y}} = \tilde{g}_i^{\tilde{Z}}$$

whence, taking into account (3.27),

$$a = (\rho(x_n) - \rho(y_n))^{-1}\rho(x_n), \qquad b = -(\rho(x_n) - \rho(y_n))^{-1}\rho(y_n) \tag{3.34}$$

The wanted vector $\tilde{y}$ such that $y \sim \tilde{y}$ and $x - y \sim \tilde{\Gamma}(\tilde{g}^{\tilde{X}}) - \tilde{y}$ is now given by

$$\tilde{y} = -a^{-1}b\tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) = (\rho(x_n))^{-1}\rho(y_n)\tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) \tag{3.35}$$

Let now u^1, u^2 and u^3 be three independent vectors of V_n and let $\tilde{u}^1 \in V_n^*$ be such that $u^1 \sim \tilde{u}^1$. Setting $\tilde{u}^2 = T_{u^1,\tilde{u}^1}(u^2)$ and $\tilde{u}^3 = T_{u^1,\tilde{u}^1}(u^3)$ the following map can be defined (Varadarajan, 1968, Lemma 3.9)

$$L : V_n \to V_n^*,$$

$$Lx = \begin{cases} 0, & \text{if } x = 0, \\ T_{u^l,\tilde{u}^l}(x) & \text{with } l \in \{1, 2, 3\} \text{ such that } \mathbb{D}.x \neq \mathbb{D}.u^l, \text{ if } x \neq 0 \end{cases} \tag{3.36}$$

For $x \neq 0$ the relation $x \sim Lx$ holds. Thus the equation

$$L(cx) = g(c, x)Lx \tag{3.37}$$

holds for any $x \neq 0$ and any $c \in \mathbb{D}$, where $g(c, x) \in \mathbb{D}^0$ is such that $g(0, x) = 0^0$. In Varadarajan (1968, Lemmas 3.12 and 3.13) it is shown that in fact $g(c, x)$ does not depend on x and that the map

$$\sigma : \mathbb{D} \to \mathbb{D}^0, \qquad \sigma(c) = g(c, x) \tag{3.38}$$

where x is any non-zero vector of V_n, is an isomorphism of $\mathbb{D}$ onto $\mathbb{D}^0$.

We take now u^1, u^2 and u^3 such that $u_n{}^l \neq 0, l \in \{1, 2, 3\}$, and $\rho(u_n{}^1) = 1^0$ and take as a vector $\tilde{u}^1$ related to u^1 the vector $\tilde{u}^1 = \tilde{\Gamma}(\tilde{g}^{\tilde{U}^1})$, where $\tilde{U}^1 = \eta(\mathbb{D}.u^1)$. Moreover we choose for the vector x in (3.37) a vector y such that $\mathbb{D}.y \neq \mathbb{D}.u^1$, $y_n \neq 0$ and $y_n \neq u_n{}^1$. Then from (3.13)-(3.26) we have

$$Ly = \rho(y_n)\tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) \tag{3.39}$$

and, for any $c \neq y_n^{-1}$,

$$L(cy) = \rho(cy_n)\tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) = \rho(c)\rho(y_n)\tilde{\Gamma}(\tilde{g}^{\tilde{Y}}) \tag{3.40}$$

Therefore, from (3.27) we obtain

$$g(c, x) = \rho(c), \qquad c \neq y_n^{-1} \tag{3.41}$$

Since ρ is continuous, from (3.41) it follows that the isomorphism σ is continuous and then that $\sigma(c) = \rho(c)$ for every $c \in \mathbb{D}$.

From Varadarajan (1968, Theorems 3.1, 4.1, 4.5, 4.6 and 7.40) it follows that the antiautomorphism θ is given by $\theta(c) = d\rho(c)d^{-1}$, where d is a suitable non-zero element of $\mathbb{D}$ (obviously ρ is now regarded as an antiautomorphism of $\mathbb{D}$). Then we can conclude that the antiautomorphism θ is continuous.

This completes the proof of the theorem.

Appendix

Let $\mathscr{L}$ be any logic and $\mathscr{P}$ the set of pure states of $\mathscr{L}$. The 'topology of states' is defined in the following way.

Given any net $\{a_\alpha\}_{\alpha \in A}$ in $\mathscr{L}$ we say that $\{a_\alpha\}$'converges' to $a \in \mathscr{L}$, and write $a_\alpha \to a$, if for every $s \in \mathscr{P}$ the net $\{s(a_\alpha)\}$ converges to $s(a)$ in the usual topology of $\mathbb{R}$ and take as the family of closed sets the family of the subsets N of $\mathscr{L}$ which satisfy the condition: $\{a_\alpha\}$ is a net in N and $a_\alpha \to a$ imply $a \in N$.

The following important theorem can be proved.

*Theorem A.*1. Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be two logics and $\xi : \mathscr{L} \to \tilde{\mathscr{L}}$ an isomorphism. If on $\mathscr{L}$ the topology of states is introduced, the quotient topology on $\tilde{\mathscr{L}}$ relative to ξ and to the topology of $\mathscr{L}$ is the topology of states on $\tilde{\mathscr{L}}$.

Proof. Let $\mathscr{P}$ and $\tilde{\mathscr{P}}$ be the set of pure states on $\mathscr{L}$ and $\tilde{\mathscr{L}}$, respectively. For any $s \in \mathscr{P}$ let $\tilde{s} = s \circ \xi^{-1}$. The correspondence $s \rightsquigarrow \tilde{s}$ obviously is a bijection from $\mathscr{P}$ onto $\tilde{\mathscr{P}}$.

Let us introduce in $\tilde{\mathscr{L}}$ the quotient topology relative to ξ and to the topology of states on $\mathscr{L}$. Then ξ and ξ^{-1} are continuous maps (Kelley, 1955, page 94). We can prove that every subset of $\tilde{\mathscr{L}}$ closed in the quotient topology of $\tilde{\mathscr{L}}$ is closed also in the topology of states of $\tilde{\mathscr{L}}$ and vice versa.

If $\tilde{B}$ is a subset of $\tilde{\mathscr{L}}$ closed in the quotient topology then $B = \xi^{-1}(\tilde{B})$ is closed in $\mathscr{L}$. Let $\{\tilde{a}_\alpha\}$ be a net in $\tilde{B}$ converging to $\tilde{a} \in \tilde{\mathscr{L}}$, namely such that $\tilde{s}(\tilde{a}_\alpha) \to \tilde{s}(\tilde{a})$, $\forall \tilde{s} \in \tilde{\mathscr{P}}$. Then $\{a_\alpha\}$, where $a_\alpha = \xi^{-1}(\tilde{a}_\alpha)$, is a net in B converging to $a = \xi^{-1}(\tilde{a})$ since for any $s \in \mathscr{P}$ we can write $s = \tilde{s} \circ \xi$ and thus we have

$$s(a_\alpha) = \tilde{s}(\tilde{a}_\alpha) \to \tilde{s}(\tilde{a}) = s(a), \qquad \forall s \in \mathscr{P}$$

Since B is closed, a belongs to B. Therefore $\tilde{a}$ belongs to $\tilde{B}$ and this shows that $\tilde{B}$ is closed in the topology of states.

Conversely if $\tilde{E}$ is a subset of $\tilde{\mathscr{L}}$ closed in the topology of states then, setting $E = \xi^{-1}(\tilde{E})$, for every net $\{a_\alpha\}$ in E converging to $a \in \mathscr{L}$ we have that $\{\tilde{a}_\alpha\}$, where $\tilde{a}_\alpha = \xi(a_\alpha)$, is a net in $\tilde{E}$ converging to $\tilde{a} = \xi(a)$ since for any $\tilde{s} \in \tilde{\mathscr{P}}$ we can write $\tilde{s} = s \circ \xi^{-1}$ and thus we have

$$\tilde{s}(\tilde{a}_\alpha) = s(a_\alpha) \to s(a) = \tilde{s}(\tilde{a}_\alpha), \qquad \forall \tilde{s} \in \tilde{\mathscr{P}}$$

Therefore $a \in E$ and E is closed, namely $\tilde{E}$ is closed in the quotient topology.

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